

# Structure of states saturating an another version of strong subadditivity

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## Abstract

Strong subadditivity inequality of quantum entropy, proved by Lieb and Ruskai, is a powerful tool in quantum information theory. The fact that the strong subadditivity inequality saturated only by a so-called Markov states is obtained in the recent literature [P. Hayden *et al.*, Commun. Math. Phys. **246**, 359 (2004).].

In this report, we will give a characterization to an another equivalent version of strong subadditivity inequality. The possible applications are discussed.

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## 1 Introduction and preliminaries

The celebrated strong subadditivity (SSA) inequality of quantum entropy, proved by Lie and Ruskai in [1],

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC}), \quad (1.1)$$

is a very powerful tool in quantum information theory. An another equivalent version of SSA can be described as follows:

$$S(\sigma_A) + S(\sigma_C) \leq S(\sigma_{AB}) + S(\sigma_{CB}). \quad (1.2)$$

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In fact, SSA plays an essential role in nearly every nontrivial insight in quantum information theory, for instance, as some direct consequences of SSA, the data processing inequality, the well-known Holevo bound [2] and the properties of the coherent information [3], etc. Moreover, SSA connects with the monotonicity of relative entropy under quantum channels [4], i.e., they can imply each other.

The condition of equality is an interesting and important subject. An extremely important result, i.e., the structure of states which satisfy Eq. (1.1) with equality, was obtained in [5]. An analogous characterization for its equivalent version, that is Eq. (1.2) with equality, is highly desirable. The present investigation aims to resolve this problem.

To begin with, we fix notation that will be used in this context. Let  $\mathcal{H}$  be a finite dimensional complex Hilbert space. A *quantum state*  $\rho$  on  $\mathcal{H}$  is a positive semi-definite operator of trace one, in particular, for each unit vector  $|\psi\rangle \in \mathcal{H}$ , the operator  $\rho = |\psi\rangle\langle\psi|$  is said to be a *pure state*. The set of all quantum states on  $\mathcal{H}$  is denoted by  $D(\mathcal{H})$ . For each quantum state  $\rho \in D(\mathcal{H})$ , its von Neumann entropy is defined by  $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$ . A *quantum operation*  $\Phi$  on  $\mathcal{H}$  is a trace-preserving completely positive linear mapping defined over the set  $D(\mathcal{H})$ . It follows from ([6, Prop. 5.2 and Cor. 5.5]) that there exists linear operators  $\{K_\mu\}_\mu$  on  $\mathcal{H}$  such that  $\sum_\mu K_\mu^\dagger K_\mu = \mathbb{1}$  and  $\Phi = \sum_\mu \text{Ad}_{K_\mu}$ , that is, for each quantum state  $\rho$ , we have the Kraus representation

$$\Phi(\rho) = \sum_\mu K_\mu \rho K_\mu^\dagger.$$

The corresponding complementary channel  $\hat{\Phi}$  for  $\Phi$  is defined as

$$\hat{\Phi}(\rho) = \sum_{\mu, \nu} \text{Tr}(K_\mu \rho K_\nu^\dagger) |\mu\rangle\langle\nu|.$$

Let  $\mathcal{E} = \{(p_\mu, \rho_\mu)\}$  be a quantum ensemble on  $\mathcal{H}$ , that is, each  $\rho_\mu \in D(\mathcal{H})$ ,  $p_\mu > 0$ , and  $\sum_\mu p_\mu = 1$ . The *Holevo quantity* of the quantum ensemble  $\{(p_\mu, \rho_\mu)\}$  is defined by the following expression:

$$\chi \{(p_\mu, \rho_\mu)\} = S(\sum_\mu p_\mu \rho_\mu) - \sum_\mu p_\mu S(\rho_\mu). \quad (1.3)$$

## 2 Main result and applications

In this section, we give a characterization to the structure of states saturating an another version of strong subadditivity of quantum entropy. The potential applications are presented.

## 2.1 The saturation of an equivalent version of SSA

**Theorem 2.1.** *Let  $\sigma_{ABC} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ . Then*

$$S(\sigma_A) + S(\sigma_C) = S(\sigma_{AB}) + S(\sigma_{CB}) \quad (2.1)$$

*if and only if there are two decompositions of system  $A$  and  $C$ , respectively, as*

$$\mathcal{H}_A = \bigoplus_{i=1}^{K_A} \mathcal{H}_{a_i^L} \otimes \mathcal{H}_{a_i^R} \quad \text{and} \quad \mathcal{H}_C = \bigoplus_{j=1}^{K_C} \mathcal{H}_{c_j^L} \otimes \mathcal{H}_{c_j^R} \quad (2.2)$$

*such that*

$$\sigma_{ABC} = \bigoplus_{i,j} \mu_{ij} \sigma_{a_i^L B c_j^L} \otimes \sigma_{a_i^R c_j^R}, \quad (2.3)$$

*where  $\sigma_{a_i^L B c_j^L} \equiv |\psi\rangle\langle\psi|_{a_i^L B c_j^L} \in \mathcal{D}(\mathcal{H}_{a_i^L} \otimes \mathcal{H}_B \otimes \mathcal{H}_{c_j^L})$ ,  $\sigma_{a_i^R c_j^R} \in \mathcal{D}(\mathcal{H}_{a_i^R} \otimes \mathcal{H}_{c_j^R})$  and  $\{\mu_{ij}\}$  is a joint probability distribution.*

*Proof.* We introduce a reference system  $D$  such that  $\sigma_{ABCD}$  is a purification of  $\sigma_{ABC}$ . Thus Eq. (2.1) can be rewritten as

$$S(\sigma_A) + S(\sigma_C) = S(\sigma_{CD}) + S(\sigma_{AD}). \quad (2.4)$$

It can be seen that, when the systems  $A$  and  $C$  are fixed, the systems  $B$  and  $D$  play a symmetric role in Eq. (2.1) and Eq. (2.4). Analogously, we have

$$S(\sigma_A) + S(\sigma_{ABD}) = S(\sigma_{AB}) + S(\sigma_{AD}), \quad (2.5)$$

$$S(\sigma_{CBD}) + S(\sigma_C) = S(\sigma_{CD}) + S(\sigma_{CB}). \quad (2.6)$$

Again, when the systems  $B$  and  $D$  are fixed, the systems  $A$  and  $C$  play a symmetric role in Eq. (2.5) and Eq. (2.6).

Now it follows from Lemma 3.1 that there are two decompositions of  $A$  and  $C$ , respectively,

$$\mathcal{H}_A = \bigoplus_{i=1}^{K_A} \mathcal{H}_{a_i^L} \otimes \mathcal{H}_{a_i^R} \quad \text{and} \quad \mathcal{H}_C = \bigoplus_{j=1}^{K_C} \mathcal{H}_{c_j^L} \otimes \mathcal{H}_{c_j^R} \quad (2.7)$$

such that

$$\sigma_{ABD} = \bigoplus_i p_i \sigma_{a_i^L B} \otimes \sigma_{a_i^R D} \quad \text{and} \quad \sigma_{BCD} = \bigoplus_j q_j \sigma_{B c_j^L} \otimes \sigma_{c_j^R D}. \quad (2.8)$$

Thus  $\sigma_{ABC}$  must be of the form:

$$\sigma_{ABC} = \bigoplus_{i,j} \mu_{ij} \sigma_{a_i^L B c_j^L}^{(ij)} \otimes \sigma_{a_i^R c_j^R}^{(ij)},$$

where

$$S\left(\sigma_{a_i^L B}^{(ij)}\right) + S\left(\sigma_{B c_j^L}^{(ij)}\right) = S\left(\sigma_{a_i^L}^{(ij)}\right) + S\left(\sigma_{c_j^L}^{(ij)}\right) \quad (\forall i, j).$$

Without loss of generality, we assume that the system  $a_i^L$  and  $c_j^L$  can not be decomposed like the  $\mathcal{H}_A$  and  $\mathcal{H}_C$ , respectively. Therefore  $\sigma_{a_i^L B c_j^L}$  must be a pure state, which implies that

$$\sigma_{a_i^L B c_j^L} \equiv |\psi\rangle\langle\psi|_{a_i^L B c_j^L}.$$

If the state  $\sigma_{ABC}$  is of the form Eq. (2.3), then it is easy to check that Eq. (2.1) holds.  $\square$

**Remark 2.2.** A simple example for Eq. (2.1) is a pure tripartite state  $\sigma_{ABC}$ . In the above proof, based on this point, by employing Proposition 3.1, we obtain that there must exist a decomposition of  $\sigma_{ABC}$  such that its substates are locally pure states.

## 2.2 Applications

In the following, we make a theoretical analysis of Roga's result concerning universal bound for Holevo information [2].

Consider a state  $\rho$ , a quantum channel  $\Phi$ , and the image of  $\rho$  under  $\Phi$ :

$$\rho' = \Phi(\rho) = \sum_{\mu} K_{\mu} \rho K_{\mu}^{\dagger}. \quad (2.9)$$

The complementary channel produces a *correlation matrix*

$$\widehat{\Phi}(\rho) = \sum_{\mu, \nu} \text{Tr} \left( K_{\mu} \rho K_{\nu}^{\dagger} \right) |\mu\rangle\langle\nu|. \quad (2.10)$$

Denote

$$q_{\mu} = \text{Tr} \left( K_{\mu} \rho K_{\mu}^{\dagger} \right) \quad \text{and} \quad \rho'_{\mu} = q_{\mu}^{-1} K_{\mu} \rho K_{\mu}^{\dagger}$$

so that

$$\rho' = \sum_{\mu} q_{\mu} \rho'_{\mu}.$$

Then the Holevo information is bounded by the *exchange entropy*:

$$\chi(\{q_{\mu}, \rho'_{\mu}\}) \leq S\left(\widehat{\Phi}(\rho)\right). \quad (2.11)$$

Moreover, the *average entropy* is bounded by the entropy of the initial state:

$$\sum_{\mu} q_{\mu} S(\rho'_{\mu}) \leq S(\rho). \quad (2.12)$$

Now in order to make an analysis of the saturations in Eq. (2.11) and Eq. (2.12), we need to go back to the original proof. In the proof of the above inequalities, the authors in [2] introduced a tripartite

$$\omega_{ABC} \stackrel{\text{def}}{=} \sum_{\mu, \nu} \left( K_{\mu} \rho K_{\nu}^{\dagger} \right)_A \otimes |\mu\rangle\langle\nu|_B \otimes |\mu\rangle\langle\nu|_C. \quad (2.13)$$

From the above expression, we see that  $\omega_{ABC}$  is a symmetric state with respect to  $BC$  and  $\mathcal{H}_B = \mathcal{H}_C$ . It is convenient to introduce the notation  $A_{\mu\nu} = K_{\mu} \rho K_{\nu}^{\dagger}$ , so that  $q_{\mu} = \text{Tr} (A_{\mu\mu})$  and  $\rho'_{\mu} = q_{\mu}^{-1} A_{\mu\mu}$ . Since

$$S(\omega_{BC}) = S(\widehat{\Phi}(\rho)), \quad S(\omega_A) = S\left(\sum_{\mu} q_{\mu} \rho'_{\mu}\right), \quad \sum_{\mu} q_{\mu} S(\rho'_{\mu}) = S(\omega_{AC}) - S(\omega_B).$$

Therefore

$$\chi(\{q_{\mu}, \rho'_{\mu}\}) = S\left(\widehat{\Phi}(\rho)\right) \iff S(\omega_A) + S(\omega_C) = S(\omega_{AB}) + S(\omega_{BC}).$$

This amounts to say, by Theorem 2.1, that

$$\omega_{ABC} = \sum_{\mu, \nu} \left( K_{\mu} \rho K_{\nu}^{\dagger} \right)_A \otimes |\mu\rangle\langle\nu|_B \otimes |\mu\rangle\langle\nu|_C = \bigoplus_{i,j} p_{ij} \omega_{a_i^L B c_j^L}^{(ij)} \otimes \omega_{a_i^R c_j^R}^{(ij)}, \quad (2.14)$$

where each  $\omega_{a_i^L B c_j^L}^{(ij)}$  is a pure state. Since both  $B$  and  $C$  are identical, it follows that

$$\sum_{\mu} K_{\mu} \rho K_{\mu}^{\dagger} \otimes |\mu\rangle\langle\mu| = \bigoplus_{i,j} p_{ij} \omega_{a_i^L c_j^L}^{(ij)} \otimes \omega_{a_i^R c_j^R}^{(ij)},$$

which implies that

$$\Phi(\rho) = \omega_A = \bigoplus_{i,j} p_{ij} \omega_{a_i^L}^{(ij)} \otimes \omega_{a_i^R}^{(ij)}.$$

Moreover

$$\sum_{\mu} q_{\mu} S(\rho'_{\mu}) = S(\rho) \iff S(\omega_{AC}) - S(\omega_B) = S(\omega_{ABC}) = S(\omega_{AB}) - S(\omega_C).$$

This amounts to say, by Proposition 3.3, that

$$\omega_{ABC} = \omega_L \otimes |\psi\rangle\langle\psi|_{RC}, \quad \omega_{ACB} = \omega_L \otimes |\psi\rangle\langle\psi|_{RB},$$

which implies that

$$\omega_A = \omega_L = \omega_{\hat{L}}, \quad \omega_R = \omega_B, \quad \omega_C = \omega_{\hat{R}}.$$

Furthermore,  $|\psi\rangle\langle\psi|_{BC}$  is a symmetric state on  $\mathcal{H}_B \otimes \mathcal{H}_C$  with  $\mathcal{H}_B = \mathcal{H}_C$ . Now we have

$$|\psi\rangle\langle\psi|_{BC} = \sum_{\mu,\nu} \text{Tr} \left( K_\mu \rho K_\nu^\dagger \right) |\mu\rangle\langle\nu| \otimes |\mu\rangle\langle\nu|.$$

This indicates that  $\sum_{\mu,\nu} \text{Tr} \left( K_\mu \rho K_\nu^\dagger \right) |\mu\rangle\langle\nu| \equiv \hat{\Phi}(\rho)$  is still a pure state. Therefore

$$\sum_{\mu,\nu} K_\mu \rho K_\nu^\dagger \otimes |\mu\rangle\langle\nu| = \Phi(\rho) \otimes \hat{\Phi}(\rho). \quad (2.15)$$

Let  $\text{Tr} (K_\mu \rho K_\nu^\dagger) = \lambda_\mu \bar{\lambda}_\nu$  for complex numbers  $\lambda_\mu$ . Then  $\sum_\mu |\lambda_\mu|^2 = 1$ . Now we can infer from Eq. (2.15) that

$$\Phi(\rho) = \left( \lambda_\mu^{-1} K_\mu \right) \rho \left( \lambda_\nu^{-1} K_\nu \right)^\dagger = \left( \lambda_\nu^{-1} K_\nu \right) \rho \left( \lambda_\mu^{-1} K_\mu \right)^\dagger \quad (\forall \mu, \nu)$$

or

$$K_\mu \rho K_\nu^\dagger = \lambda_\mu \bar{\lambda}_\nu \Phi(\rho),$$

which implies that

$$\rho = \left( \sum_\mu K_\mu^\dagger K_\mu \right) \rho \left( \sum_\nu K_\nu^\dagger K_\nu \right) = \sum_{\mu,\nu} K_\mu^\dagger (K_\mu \rho K_\nu^\dagger) K_\nu \quad (2.16)$$

$$= \left( \sum_\mu \lambda_\mu K_\mu^\dagger \right) \rho \left( \sum_\nu \lambda_\nu K_\nu^\dagger \right)^\dagger \equiv M \Phi(\rho) M^\dagger, \quad (2.17)$$

where  $M \stackrel{\text{def}}{=} \sum_\mu \lambda_\mu K_\mu^\dagger$ .

### 3 Appendix: A list of results related to SSA with equality

**Proposition 3.1** ([5]). *A state  $\rho_{ABC} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$  saturating the strong subadditivity inequality, i.e.,*

$$S(\rho_{AB}) + S(\rho_{BC}) = S(\rho_{ABC}) + S(\rho_B) \quad (3.1)$$

*if and only if there is a decomposition of system B as*

$$\mathcal{H}_B = \bigoplus_j \mathcal{H}_{b_j^L} \otimes \mathcal{H}_{b_j^R} \quad (3.2)$$

into a direct (orthogonal) sum of tensor products, such that

$$\rho_{ABC} = \bigoplus_j \lambda_j \rho_{Ab_j^L} \otimes \rho_{b_j^R C}, \quad (3.3)$$

where  $\rho_{Ab_j^L} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_{b_j^L})$  and  $\rho_{b_j^R C} \in \mathcal{D}(\mathcal{H}_{b_j^R} \otimes \mathcal{H}_C)$ , and  $\{\lambda_j\}$  is a probability distribution.

**Proposition 3.2** ([7, 8]). *Let  $\rho_{ABC} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$  for which strong subadditivity is saturated for both triples  $ABC, BAC$ . Then  $\rho_{ABC}$  must have the following form:*

$$\rho_{ABC} = \bigoplus_{i,j} p_{ij} \rho_{a_i^L}^{(i)} \otimes \rho_{a_i^R b_j^L}^{(ij)} \otimes \rho_{b_j^R}^{(j)} \otimes \rho_C^{(k)}, \quad (3.4)$$

where  $k$  is a function only of  $i, j$  in the sense that

$$k = k(i, j) = k_1(i) = k_2(j) \quad \text{whenever} \quad p_{ij} > 0.$$

In particular,  $k$  need only be defined where  $p_{ij} > 0$  so that it is not necessarily constant. By collecting the terms of equivalent  $k$  we can write

$$\rho_{ABC} = \bigoplus_k p_k \rho_{AB}^{(k)} \otimes \rho_C^{(k)}, \quad (3.5)$$

where

$$p_k \rho_{AB}^{(k)} = \sum_{(i,j): k(i,j)=k} p_{ij} \rho_{a_i^L}^{(i)} \otimes \rho_{a_i^R b_j^L}^{(ij)} \otimes \rho_{b_j^R}^{(j)}. \quad (3.6)$$

**Proposition 3.3** ([9]). *Let  $\omega_{BC} \in \mathcal{D}(\mathcal{H}_B \otimes \mathcal{H}_C)$ . Then  $S(\omega_{BC}) = S(\omega_B) - S(\omega_C)$  if and only if the following statements are valid:*

- (i)  $\mathcal{H}_B$  can be factorized into the form  $\mathcal{H}_B = \mathcal{H}_L \otimes \mathcal{H}_R$ ,
- (ii)  $\omega_{BC} = \omega_L \otimes |\psi\rangle\langle\psi|_{RC}$  for  $|\psi\rangle_{RC} \in \mathcal{H}_R \otimes \mathcal{H}_C$ , where  $\omega_L \in \mathcal{D}(\mathcal{H}_L)$ .

**Proposition 3.4** ([3]). *Let  $\rho \in \mathcal{D}(\mathcal{H})$  and  $\Phi \in \mathcal{T}(\mathcal{H})$  be a quantum channel. The coherent information, defined by*

$$I_c(\rho, \Phi) \stackrel{\text{def}}{=} S(\Phi(\rho)) - S(\widehat{\Phi}(\rho)),$$

*achieves its maximum, that is,  $I_c(\rho, \Phi) = S(\rho)$  if and only if the following statements holds:*

- (i) the underlying Hilbert space can be decomposed as:  $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$ ;
- (ii) the output state of the quantum channel  $\Phi$  is of a product form:  $\Phi(\rho) = \rho_L \otimes \rho_R$  for  $\rho_L \in \mathcal{D}(\mathcal{H}_L), \rho_R \in \mathcal{D}(\mathcal{H}_R)$ .

## Acknowledgement

We want to express our heartfelt thanks to W. Roga for useful comments. This project is supported by Natural Science Foundations of China (11171301, 10771191 and 10471124) and Natural Science Foundation of Zhejiang Province of China (Y6090105).

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